If $Q(x)$ had two roots in the interval $(0,1)$, then $Q'(x)$ would have a root in the interval $(0,1)$ (by Rolle's theorem and our remarks about roots with multiplicity), which is not the case. Similarly $Q(x)$ cannot have more than one root greater than $\frac{3}{2}$. Since $Q(x)$ has at least three positive roots, though, it has a root in the interval $[1,\frac{3}{2}]$. Looking at $Q'(x)$ we can see that $Q(x)$ is decreasing on this interval and we obtain $Q(1)\geq 0$ and therefore $p\leq\frac{5}{2}$, which finishes the proof.

The bound can be obtained by setting one variable to $\frac{5}{2}$ and the others to 1.

4098. *Proposed by Ardak Mirzakhmedov.*

Let α,β and γ be acute angles such that $\alpha+\beta=\gamma$. Show that

$$\cos\alpha+\cos\beta+\cos\gamma-1\geq 2\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}.$$

We received six correct submissions. We present the solution by Arkady Alt.

Note first that for all $a,b,c,d\in\mathbb{R}$, we have $(ac-bd)^2-(ad-bc)^2=(a^2-b^2)(c^2-d^2)$, so $(a^2-b^2)(c^2-d^2)\leq(ac-bd)^2$. In particular, if $a>b>0$ and $c>d>0$, then

$$ac-bd\geq\sqrt{a^2-b^2}\cdot\sqrt{c^2-d^2}. \qquad (1)$$

Next,

$$\begin{aligned}\cos\alpha+\cos\beta+\cos\gamma-1&=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-2\sin^2\frac{\gamma}{2}\\&=2\left(\cos\frac{\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}-\sin\frac{\gamma}{2}\cdot\sin\frac{\alpha+\beta}{2}\right). \qquad (2)\end{aligned}$$

Since $\frac{\alpha+\beta}{2}=\frac{\gamma}{2}\in(0,\frac{\pi}{4})$ we have

$$\cos\frac{\alpha-\beta}{2}>\cos\frac{\alpha+\beta}{2}=\cos\frac{\gamma}{2}>\sin\frac{\gamma}{2}.$$

Hence, if we let

$$a=\cos\frac{\gamma}{2}, b=d=\sin\frac{\gamma}{2}\quad\text{and}\quad c=\cos\frac{\alpha-\beta}{2},$$

then $a>b>0$ and $c>d>0$ so applying (1) we obtain

$$\begin{aligned}\cos\frac{\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}-\sin\frac{\gamma}{2}\cdot\sin\frac{\alpha+\beta}{2}&\geq\sqrt{\cos^2\frac{\gamma}{2}-\sin^2\frac{\gamma}{2}}\cdot\sqrt{\cos^2\frac{\alpha-\beta}{2}-\sin^2\frac{\alpha+\beta}{2}}\\&=\sqrt{\cos\gamma}\sqrt{\frac{1}{2}\left(1+\cos(\alpha-\beta)-(1-\cos(\alpha+\beta)\right)}\\&=\sqrt{\cos\gamma}\sqrt{\cos\alpha\cdot\cos\beta}\\&=\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}. \qquad (3)\end{aligned}$$

Substituting (3) into (2), we then have

$$\cos\alpha+\cos\beta+\cos\gamma-1\geq 2\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma},$$

thus completing the proof.